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Explicit formula for the Siegel series of a quadratic form over a non-archimedean local field

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1 Introduction

The Siegel series is one of the simplest but most important subjects, and it appears in the Fourier coefficients of the Hilbert-Siegel Eisenstein series (cf. [10]) and of the Duke-Imamoglu-Ikeda lift (cf. [2], [6]). Moreover it is also related with arithmetic geometry (cf. [8]). In any case, it is very important to give an explicit form of the Siegel series. In [7], the second named author gave an explicit formula for the Siegel series of a half-integral matrix over $\mathbb{Z}_p$ with any prime number p of any degree. In this report, we give an explicit formula for the Siegel series of a half-integral matrix of any degree over any non-archimedean local field of characteristic 0. This topic is discussed in detail in [5]

2 Siegel series

Let F be a non-archimedean local field of characteristic 0, and $\mathfrak{o} = \mathfrak{o}_F$ its ring of integers. The maximal ideal and the residue field of $\mathfrak{o}$ is denoted by $\mathfrak{p}$ and $\mathfrak{k}$, respectively. We fix a prime element ϖ of $\mathfrak{o}$ once and for all. The cardinality of $\mathfrak{k}$ is denoted by q. Let $\mathrm{ord} = \mathrm{ord}_{\mathfrak{p}}$ denote additive valuation on F normalized so that $\mathrm{ord}(\varpi) = 1$. If $a = 0$, We write $\mathrm{ord}(0) = \infty$ and we make the convention that $\mathrm{ord}(0) > \mathrm{ord}(b)$ for any $b \in F^{\times}$. We also denote by $|\ast|_{\mathfrak{p}}$ denote the valuation on F normalized so that $|\varpi|_{\mathfrak{p}} = q^{-1}$. We put $e_0 = \mathrm{ord}_{\mathfrak{p}}(2)$. For an integral domain R, let $\mathrm{Sym}_n(R)$ be the set of symmetric matrices of degree n with entries in R. We say that an element A of $\mathrm{Sym}_n(R)$ is non-degenerate if the determinant $\det A$ of A is non-zero. We say that a symmetric matrix $A = (a_{ij})$ of degree n with entries in F is half-integral over $\mathfrak{o}$ if a_{ii} $(i = 1, ..., n)$ and $2a_{ij}$ $(1 \le i \ne j \le n)$ belong to $\mathfrak{o}$. We denote by $\mathcal{H}_n(\mathfrak{o})$ the set of half-integral matrices of degree n over $\mathfrak{o}$, and by $\mathcal{H}_n(\mathfrak{o})^{\mathrm{nd}}$ the subset of $\mathcal{H}_n(\mathfrak{o})$ consisting of non-degenerate matrices.

For an element $B \in \mathcal{H}_n(\mathfrak{o})^{\mathrm{nd}}$, we put $D_B = (-4)^{[n/2]} \det B$. If n is even, we denote the discriminant ideal of $F(\sqrt{D_B})/F$ by $\mathfrak{D}_B$. We also put

$$\xi_B = \begin{cases} 1 & \text{if } D_B \in F^{\times 2}, \\ -1 & \text{if } F(\sqrt{D_B})/F \text{ is unramified quadratic,} \\ 0 & \text{if } F(\sqrt{D_B})/F \text{ is ramified quadratic.} \end{cases}$$

Put

$$\mathfrak{e}_B = \begin{cases} \mathrm{ord}(D_B) - \mathrm{ord}(\mathfrak{D}_B) & \text{if } n \text{ is even} \\ \mathrm{ord}(D_B) & \text{if } n \text{ is odd.} \end{cases}$$

Let $\langle\ ,\ \rangle = \langle\ ,\ \rangle_F$ be the Hilbert symbol on F. Let B be a non-degenerate symmetric matrix with entries in F of degree n. Then B is $GL_n(F)$-equivalent to $b_1 \perp \cdots \perp b_n$ with $b_1, \ldots, b_n \in F^\times$. Then we define ε_B as

$$\varepsilon_B = \prod_{1 \le i < j \le n} \langle b_i, b_j \rangle.$$

This does not depend on the choice of $b_1, \ldots, b_n$. We also denote by η_B the Clifford invariant of B (cf. [3]). Then we have

$$\eta_B = \begin{cases} \langle -1, -1 \rangle^{m(m+1)/2} \langle (-1)^m, \det B \rangle \varepsilon_B & \text{if } n = 2m+1 \\ \langle -1, -1 \rangle^{m(m-1)/2} \langle (-1)^{m+1}, \det B \rangle \varepsilon_B & \text{if } n = 2m. \end{cases}$$

(cf. [[3], Lemma 2.1]). We make the convention that $\xi_B = 1, \mathfrak{e}_B = 0$ and $\eta_B = 1$ if B is the empty matrix. Once for all, we fix an additive character ψ of F of order zero, that is, a character such that

$$\mathfrak{o} = \{a \in F \mid \psi(ax) = 1 \ \text{ for any } x \in \mathfrak{o}\}.$$

For a half-integral matrix B of degree n over $\mathfrak{o}$ define the local Siegel series $b_\mathfrak{p}(B, s)$ by

$$b_\mathfrak{p}(B, s) = \sum_R \psi(\mathrm{tr}(BR)) \mu(R)^{-s},$$

where R runs over a complete set of representatives of $\mathrm{Sym}_n(F)/\mathrm{Sym}_n(\mathfrak{o})$ and $\mu(R) = [R\mathfrak{o}^n + \mathfrak{o}^n : \mathfrak{o}^n]$.

Now for a non-degenerate half-integral matrix B of degree n over $\mathfrak{o}$ define a polynomial $\gamma_q(B, X)$ in X by

$$\gamma_q(B, X) = \begin{cases} (1-X) \prod_{i=1}^{n/2} (1 - q^{2i} X^2)(1 - q^{n/2} \xi_B X)^{-1} & \text{if } n \text{ is even} \\ (1-X) \prod_{i=1}^{(n-1)/2} (1 - q^{2i} X^2) & \text{if } n \text{ is odd.} \end{cases}$$

Then it is shown by [10] that there exists a polynomial $F_\mathfrak{p}(B, X)$ in X such that

$$F_\mathfrak{p}(B, q^{-s}) = \frac{b_\mathfrak{p}(B, s)}{\gamma_q(B, q^{-s})}.$$

We define a symbol $X^{1/2}$ so that $(X^{1/2})^2 = X$. We define $\widetilde{F}_\mathfrak{p}(B, X)$ as

$$\widetilde{F}_\mathfrak{p}(B, X) = X^{-\mathfrak{e}_B/2} F(B, q^{-(n+1)/2} X).$$

We note that $\widetilde{F}_\mathfrak{p}(B, X) \in \mathbb{Q}[q^{1/2}][X, X^{-1}]$ if n is even, and $\widetilde{F}_\mathfrak{p}(B, X) \in \mathbb{Q}[X^{1/2}, X^{-1/2}]$ if n is odd. We sometimes write $F_\mathfrak{p}(B, X)$ and $\widetilde{F}_\mathfrak{p}(B, X)$ as $F(B, X)$ and $\widetilde{F}(B, X)$, respectively.

The following proposition is due to [[3], Theorem 4.1].

Proposition 2.1. *We have*

$$\widetilde{F}(B, X^{-1}) = \zeta_B \widetilde{F}(B, X),$$

where $\zeta_B = \eta_B$ *or 1 according as* n *is odd or even.*

3 The Extended Gross-Keating invariant

In this section, we review the definition of the Gross-Keating invariant [1], and define its extended version, in terms of which the Siegel series can be expressed. For two matrices $B, B' \in \mathcal{H}_n(\mathfrak{o})$, we sometimes write $B \sim B'$ if B and B' are $GL_n(\mathfrak{o})$-equivalent. The $GL_n(\mathfrak{o})$-equivalence class of B is denoted by $\{B\}$. Let $B = (b_{ij}) \in \mathcal{H}_n(\mathfrak{o})^{\mathrm{nd}}$. Let $S(B)$ be the set of all non-decreasing sequences $(a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n$ such that

$$\begin{aligned} &\mathrm{ord}(b_i) \geq a_i, \\ &\mathrm{ord}(2b_{ij}) \geq (a_i + a_j)/2 \qquad (1 \leq i, j \leq n). \end{aligned}$$

Set

$$S(\{B\}) = \bigcup_{B' \in \{B\}} S(B') = \bigcup_{U \in \mathrm{GL}_n(\mathfrak{o})} S(B[U]).$$

The Gross-Keating invariant (or the GK-invariant for short) $\underline{a} = (a_1, a_2, \ldots, a_n)$ of B is the greatest element of $S(\{B\})$ with respect to the lexicographic order $\succ$ on $\mathbb{Z}_{\geq 0}^n$. Here, the lexicographic order $\succ$ is, as usual, defined as follows. For $(y_1, y_2, \ldots, y_n), (z_1, z_2, \ldots, z_n) \in \mathbb{Z}_{\geq 0}^n$, let j be the largest integer such that $y_i = z_i$ for $i < j$. Then $(y_1, y_2, \ldots, y_n) \succ (z_1, z_2, \ldots, z_n)$ if $y_j > z_j$. The Gross-Keating invariant is denoted by $\mathrm{GK}(B)$. A sequence of length 0 is denoted by $\emptyset$. When B is a matrix of degree 0, we understand $\mathrm{GK}(B) = \emptyset$.

By definition, the Gross-Keating invariant $\mathrm{GK}(B)$ is determined only by the $GL_n(\mathfrak{o})$-equivalence class of B. We say that $B \in \mathcal{H}_n(\mathfrak{o})$ is an optimal form if $\mathrm{GK}(B) \in S(B)$. Let $B \in \mathcal{H}_n(\mathfrak{o})$. Then B is $GL_n(\mathfrak{o})$-equivalent to an optimal form B'. Then we say that B has an optimal decomposition B'. We say that $B \in \mathcal{H}_n(\mathfrak{o})$ is a diagonal Jordan form if B is expressed as

$$B = \varpi^{a_1} u_1 \perp \cdots \perp \varpi^{a_n} u_n$$

with $a_1 \leq \cdots \leq a_n$ and $u_1, \cdots, u_n \in \mathfrak{o}^\times$. Then, in the non-dyadic case, the diagonal Jordan form B above is optimal, and $\mathrm{GK}(B) = (a_1, \ldots, a_n)$. Therefore, the diagonal Jordan decomposition is an optimal decomposition. However, in the dyadic case, not all half-integral symmetric matrices have a diagonal Jordan decomposition, and the Jordan decomposition is not necessarily an optimal decomposition.

Definition 3.1. Let $\underline{a} = (a_1, \ldots, a_n)$ be a non-decreasing sequence of non-negative integers. Write $\underline{a}$ as

$$\underline{a} = (\underbrace{m_1, \ldots, m_1}_{n_1}, \ldots, \underbrace{m_r, \ldots, m_r}_{n_r})$$

with $m_1 < \cdots < m_r$ and $n = n_1 + \cdots + n_{r-1} + n_r$. For $s = 1, 2, \ldots, r$ put

$$n_s^* = \sum_{u=1}^{s} n_u,$$

and

$$I_s = \{n_{s-1}^* + 1, n_{s-1}^* + 2, \ldots, n_s^*\}.$$

We denote by $\mathfrak{S}_n$ the symmetric group of degree n. Recall that a permutation $\sigma \in \mathfrak{S}_n$ is an involution if $\sigma^2 = \mathrm{id}$.

Definition 3.2. For an involution $\sigma \in \mathfrak{S}_n$ and a non-decreasing sequence $\underline{a} = (a_1, \dots, a_n)$ of non-negative integers, we set

$$\begin{aligned} \mathcal{P}^0 &= \mathcal{P}^0(\sigma) = \{i \,|\, 1 \le i \le n,\ i = \sigma(i)\}, \\ \mathcal{P}^+ &= \mathcal{P}^+(\sigma) = \{i \,|\, 1 \le i \le n,\ a_i > a_{\sigma(i)}\}, \\ \mathcal{P}^- &= \mathcal{P}^-(\sigma) = \{i \,|\, 1 \le i \le n,\ a_i < a_{\sigma(i)}\}. \end{aligned}$$

We say that an involution $\sigma \in \mathfrak{S}_n$ is an $\underline{a}$-admissible involution if the following two conditions are satisfied.

(i) $\mathcal{P}^0$ has at most two elements. If $\mathcal{P}^0$ has two distinct elements i and j, then $a_i \not\equiv a_j \bmod 2$. Moreover, if $i \in I_s \cap \mathcal{P}^0$, then i is the maximal element of I_s, and

$$i = \max\{j \mid j \in \mathcal{P}^0 \cup \mathcal{P}^+, a_i \equiv a_i \bmod 2\}.$$

(ii) For $s = 1, \dots, r$, there is at most one element in $I_s \cap \mathcal{P}^-$. If $i \in I_s \cap \mathcal{P}^-$, then i is the maximal element of I_s and

$$\sigma(i) = \min\{j \in \mathcal{P}^+ \,|\, j > i,\ a_j \equiv a_i \bmod 2\}.$$

(iii) For $s = 1, \dots, r$, there is at most one element in $I_s \cap \mathcal{P}^+$. If $i \in I_s \cap \mathcal{P}^+$, then i is the minimal element of I_s and

$$\sigma(i) = \max\{j \in \mathcal{P}^- \,|\, j < i,\ a_j \equiv a_i \bmod 2\}.$$

(iv) If $a_i = a_{\sigma(i)}$, then $|i - \sigma(i)| \le 1$.

This is called a standard $\underline{a}$-admissible involution in [4], but in this paper we omit the word "standard", since we do not consider an $\underline{a}$-admissible involution which is not standard.

Definition 3.3. For $\underline{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\ge 0}$, put

$$\begin{aligned} \mathcal{M}(\underline{a}) &= \left\{ B = (b_{ij}) \in \mathcal{H}_n(\mathfrak{o}) \middle| \begin{array}{l} \mathrm{ord}(b_{ii}) \ge a_i, \\ \mathrm{ord}(2b_{ij}) \ge (a_i + a_j)/2, \end{array} (1 \le i < j \le n) \right\}, \\ \mathcal{M}^0(\underline{a}) &= \left\{ B = (b_{ij}) \in \mathcal{H}_n(\mathfrak{o}) \middle| \begin{array}{l} \mathrm{ord}(b_{ii}) > a_i, \\ \mathrm{ord}(2b_{ij}) > (a_i + a_j)/2, \end{array} (1 \le i < j \le n) \right\}. \end{aligned}$$

Definition 3.4. Let $\sigma \in \mathfrak{S}_n$ be an $\underline{a}$-admissible involution. We say that $B = (b_{ij}) \in \mathcal{M}(\underline{a})$ is a reduced form with GK-type $(\underline{a}, \sigma)$ if the following conditions are satisfied.

(1) If $i \notin \mathcal{P}^0$ and $i \le j = \sigma(i)$, then

$$\mathrm{GK}\left(\begin{pmatrix} b_{ii} & b_{ij} \\ b_{ij} & b_{jj} \end{pmatrix}\right) = (a_i, a_j).$$

Note that if the residual characteristic of F is 2, then this condition is equivalent to the following condition.

$$\begin{cases} \mathrm{ord}(2b_{i\,\sigma(i)}) = \dfrac{a_i + a_{\sigma(i)}}{2} & \text{if } i \notin \mathcal{P}^0, \\ \mathrm{ord}(b_{ii}) = a_i & \text{if } i \in \mathcal{P}^-. \end{cases}$$

(2) If $i \in \mathcal{P}^0$, then

$$\mathrm{ord}(b_{ii}) = a_i.$$

(3) If $j \neq i, \sigma(i)$, then

$$\mathrm{ord}(2b_{ij}) > \frac{a_i + a_j}{2},$$

We often say that B is a reduced form with GK-type $\underline{a}$ without mentioning σ. We formally think of a matrix of degree 0 as a reduced form with GK-type $\emptyset$.

Remark 3.1. If the residual characteristic of F is odd, then a diagonal Jordan form $\mathrm{diag}(b_1, b_2, \ldots, b_n)$ such that $\mathrm{ord}(b_i) = a_i$ $(i = 1, 2, \ldots, n)$ is a reduced form with GK-type $\underline{a}$.

The following theorems are fundamental in our theory.

Theorem 3.1. ([[4], Theorem 5.1]) *Let B be a reduced form of* GK *type $(\underline{a}, \sigma)$. Then we have* $\mathrm{GK}(B) = \underline{a}$.

Theorem 3.2. ([[4], Theorem 4.3]) *Assume that* $\mathrm{GK}(B) = \underline{a}$ *for $B \in \mathcal{H}_n(\mathfrak{o})^{\mathrm{nd}}$. Then B is $GL_n(\mathfrak{o})$-equivalent to a reduced form of* GK *type $(\underline{a}, \sigma)$ for some $\underline{a}$-admissible involution σ.*

By Theorem 3.2, any non-degenerate half-integral symmetric matrix B over $\mathfrak{o}$ is $GL_n(\mathfrak{o})$-equivalent to a reduced form B'. Then we say that B has a reduced decomposition B'. For a matrix $C = (c_{ij})_{1 \leq i,j \leq n}$ and a positive integer $m \leq n$ we put $C^{(m)} = (c_{ij})_{1 \leq i,j \leq m}$.

Definition 3.5. Let $B \in \mathcal{H}_n(\mathfrak{o})^{\mathrm{nd}}$ with $\mathrm{GK}(B) = (a_1, \ldots, a_n)$, and $n_1, \ldots, n_r, n_1^*, \ldots, n_r^*$ and $m_1, \ldots, m_r$ be those in Definition 3.1. Take an optimal decomposition C of B, and for $s = 1, \ldots, r$ we put

$$\zeta_s(C) = \zeta(C^{(n_s^*)}),$$

where $\zeta(C^{(n_s^*)}) = \xi_{C^{(n_s^*)}}$ or $\zeta(C^{(n_s^*)}) = \eta_{C^{(n_s^*)}}$ according as n_s^* is even or odd. Then $\zeta_s(C)$ does not depend on the choice of C (cf. [[4], Theorem 0.4]), which will be denoted by $\zeta_s = \zeta_s(B)$. Then we define $\mathrm{EGK}(B)$ as $\mathrm{EGK}(B) = (n_1, \ldots, n_r; m_1, \ldots, m_r; \zeta_1, \ldots, \zeta_r)$, and we call it the extended Gross-Keating invariant of B.

4 EGK datum and its associated polynomial

To formulate our main result, we introduce an EGK datum, which is obtained by axiomatizing properties of the extended GK invariant, and attach a Laurent polynomial to it.

Definition 4.1. Let $G = (n_1, \ldots, n_r; m_1, \ldots, m_r; \zeta_1, \ldots, \zeta_r)$ be an element of $\mathbb{Z}_{>0}^r \times \mathbb{Z}_{\geq 0}^r \times \mathcal{Z}_3^r$. Put $n_s^* = \sum_{i=1}^s n_i$ for $s \leq r$. We say that G is an EGK datum of length n if the following conditions hold:

(E1) $n_r^* = n$ and $m_1 < \cdots < m_r$.

(E2) Assume that n_s^* is even. Then $\zeta_s \neq 0$ if and only if $m_1 n_1 + \cdots + m_s n_s$ is even.

(E3) Assume that n_s^* is odd. Then $\zeta_s \neq 0$. Moreover we have

(a) Assume that n_i^* is even for any $i < s$. Then

$$\zeta_s = \zeta_{s-1}^{m_s + m_{s-1}} \cdots \zeta_2^{m_2 + m_1} \zeta_1^{m_2 + m_1}.$$

In particular, $\zeta_1 = 1$ if n_1 is odd.

(b) Assume that $n_1 m_1 + \cdots + (n_{s-1} - 1) m_{s-1}$ is even and that n_i^* is odd for some $i < s$. Let $t < s$ be the largest number such that n_t^* is odd. Then

$$\zeta_s = \zeta_{s-1}^{m_s + m_{s-1}} \cdots \zeta_{t+2}^{m_{t+3} + m_{t+2}} \zeta_{t+1}^{m_{t+2} + m_{t+1}} \zeta_t.$$

In particular, $\zeta_s = \zeta_t$ if $t = s - 1$.

We denote by $\mathcal{EGK}_n$ the set of all EGK data of length n.

By construction we easily see the following.

Theorem 4.1. (cf. [[4], Theorem 6.1]) *Let $B \in \mathcal{H}_n(\mathfrak{o})^{\mathrm{nd}}$. Then* $\mathrm{EGK}(B)$ *is an* EGK *datum of length n.*

We also introduce a naive EGK datum (cf. [4]). Let $\mathcal{Z}_3 = \{0, 1, -1\}$.

Definition 4.2. An element $(a_1, \ldots, a_n; \varepsilon_1, \ldots, \varepsilon_n)$ of $\mathbb{Z}_{\geq 0}^n \times \mathcal{Z}_3^n$ is said to be a naive EGK datum of length n if the following conditions hold:

(N1) $a_1 \leq \cdots \leq a_n$.

(N2) Assume that i is even. Then $\varepsilon_i \neq 0$ if and only if $a_1 + \cdots + a_i$ is even.

(N3) Assume that i is odd. Then $\varepsilon_i \neq 0$.

(N4) $\varepsilon_1 = 1$.

(N5) Let $i \geq 3$ be an odd integer and assume that $a_1 + \cdots + a_{i-1}$ is even. Then $\varepsilon_i = \varepsilon_{i-1}^{a_i + a_{i-1}} \varepsilon_{i-2}$.

We denote by $\mathcal{NEGK}_n$ the set of all naive EGK data of length n. We will give examples of naive EGK data in Section 6.

For integers $e, \widetilde{e}$, a real number ξ, and $i = 0, 1$ define rational functions $C(e, \widetilde{e}, \xi; Y, X)$ and $D(e, \widetilde{e}, \xi; Y, X)$ in $Y^{1/2}$ and $X^{1/2}$ by

$$C(e, \widetilde{e}, \xi; Y, X) = \frac{Y^{\widetilde{e}/2} X^{-(e-\widetilde{e})/2-1}(1 - \xi Y^{-1} X)}{X^{-1} - X}$$

and

$$D(e, \widetilde{e}, \xi; Y, X) = \frac{Y^{\widetilde{e}/2} X^{-(e-\widetilde{e})/2}}{1 - \xi X}.$$

For a positive integer i put

$$C_i(e, \widetilde{e}, \xi; Y, X) = \begin{cases} C(e, \widetilde{e}, \xi; Y, X) & \text{if } i \text{ is even} \\ D(e, \widetilde{e}, \xi; Y, X) & \text{if } i \text{ is odd.} \end{cases}$$

For a sequence $\underline{a} = (a_1, \ldots, a_n)$ of integers and an integer $1 \leq i \leq n$, we define $\mathfrak{e}_i = \mathfrak{e}_i(\underline{a})$ as

$$\mathfrak{e}_i = \begin{cases} a_1 + \cdots + a_i & \text{if } i \text{ is odd} \\ 2[(a_1 + \cdots + a_i)/2] & \text{if } i \text{ is even.} \end{cases}$$

We also put $\mathfrak{e}_0 = 0$.

Definition 4.3. For a naive EGK datum $H = (a_1, \ldots, a_n; \varepsilon_1, \ldots, \varepsilon_n)$ we define a rational function $\mathcal{F}(H; Y, X)$ in $X^{1/2}$ and $Y^{1/2}$ as follows: First we define

$$\mathcal{F}(H; Y, X) = X^{-a_1/2} + X^{-a_1/2+1} + \cdots + X^{a_1/2-1} + X^{a_1/2}$$

if $n = 1$. Let $n > 1$. Then $H' = (a_1, \ldots, a_{n-1}; \varepsilon_1, \ldots, \varepsilon_{n-1})$ is a naive EGK datum of length $n - 1$. Assume that $\mathcal{F}(H'; Y, X)$ is defined for H'. Then, we define $\mathcal{F}(H; Y, X)$ as

$$\begin{aligned} \mathcal{F}(H; Y, X) &= C_n(\mathfrak{e}_n, \mathfrak{e}_{n-1}, \xi; Y, X) \mathcal{F}(H'; Y, YX) \\ &+ \zeta C_n(\mathfrak{e}_n, \mathfrak{e}_{n-1}, \xi; Y, X^{-1}) \mathcal{F}(H'; Y, YX^{-1}), \end{aligned}$$

where $\xi = \varepsilon_n$ or ε_{n-1} according as n is even or odd, and $\zeta = 1$ or ε_n according as n is even or odd.

By the definition of $\mathcal{F}(H; Y, X)$ we easily give an explicit formula for $\mathcal{F}(H; Y, X)$.

Proposition 4.1. *Let $H = (a_1, \ldots, a_n; \varepsilon_1, \ldots, \varepsilon_n)$ be a naive* EGK *datum of length n. Then we have*

$$\begin{aligned} &\mathcal{F}(H; Y, X) \\ &= \sum_{(i_1, \ldots, i_n) \in \{\pm 1\}^n} \eta_n^{(1-i_n)/2} C_n(\mathfrak{e}_n, \mathfrak{e}_{n-1}, \xi_n; Y, X^{i_n}) \\ &\times \prod_{j=1}^{n-1} \eta_j^{(1-i_j)/2} C_j(\mathfrak{e}_j, \mathfrak{e}_{j-1}, \xi_j; Y, Y^{i_j + i_j i_{j+1} + \cdots + i_j i_{j+1} \cdots i_{n-1}} X^{i_j \cdots i_n}), \end{aligned}$$

where

$$\xi_j = \begin{cases} \varepsilon_j & \text{if } j \text{ is even} \\ \varepsilon_{j-1} & \text{if } j \text{ is odd,} \end{cases}$$

and

$$\eta_j = \begin{cases} 1 & \text{if } j \text{ is even} \\ \varepsilon_j & \text{if } j \text{ is odd} \end{cases}$$

for $1 \le j \le n$. In particular,

$$\mathcal{F}(H;Y,X^{-1}) = \eta_n \mathcal{F}(H;Y,X).$$

We also have the following induction formulas.

Proposition 4.2. *Let $H = (a_1,\dots,a_n;\varepsilon_1,\dots,\varepsilon_n)$ be a naive* EGK *datum of length n and $H'' = (a_1,\dots,a_{n-2};\varepsilon_1,\dots,\varepsilon_{n-2})$. Then H'' is a naive* EGK *datum of length $n-2$. Assume that $a_{n-1} = a_n$. Then the following assertions hold.*

(1) Assume that n is odd and $a_1 + \cdots + a_{n-1}$ is even. Then we have

$$\begin{aligned}\mathcal{F}(H;Y,X) &= Y^{\mathfrak{e}_{n-2}-1} \\ &\times \left\{ \frac{X^{(-\mathfrak{e}_n+\mathfrak{e}_{n-2})/2-1}}{(YX)^{-1}-YX}\mathcal{F}(H'';Y,Y^2X) + \frac{\varepsilon_n X^{(\mathfrak{e}_n-\mathfrak{e}_{n-2})/2+1}}{(YX^{-1})^{-1}-YX^{-1}}\mathcal{F}(H'';Y,Y^2X^{-1}) \right\} \\ &+ \frac{Y^{\mathfrak{e}_{n-1}}(Y^2-Y^{-2})\varepsilon_n}{((YX)^{-1}-YX)((YX^{-1})^{-1}-YX^{-1})}\mathcal{F}(H'';Y,X).\end{aligned}$$

In particular, $\mathcal{F}(H;Y,X)$ does not depend on ε_{n-1}.

(2) Assume that n is even and $a_1 + \cdots + a_n$ is odd. Then we have

$$\begin{aligned}&\mathcal{F}(H;Y,X) \\ &= Y^{\mathfrak{e}_{n-2}} \left\{ \frac{X^{(-\mathfrak{e}_n+\mathfrak{e}_{n-2})/2-1}}{X^{-1}-X}\mathcal{F}(H'';Y,Y^2X) + \frac{X^{(\mathfrak{e}_n-\mathfrak{e}_{n-2})/2+1}}{X-X^{-1}}\mathcal{F}(H'';Y,Y^2X^{-1}) \right\}.\end{aligned}$$

In particular, $\mathcal{F}(H;Y,X)$ does not depend on ε_{n-1}.

By definition, $\mathcal{F}(H;Y,X)$ is a rational function in $X^{1/2}$ and $Y^{1/2}$ but in fact we have the following:

Theorem 4.2. *Let $H = (a_1,\dots,a_n;\varepsilon_1,\dots,\varepsilon_n)$ be a naive* EGK *datum of length n. Then $X^{\mathfrak{e}_n/2}\mathcal{F}(H;Y,X)$ is a polynomial in X of degree $\mathfrak{e}_n$ with coefficients in $\mathbb{Q}[Y,Y^{-1}]$.*

Now let us consider the relation between an EGK datum and a naive EGK datum. Let $H = (a_1,\dots,a_n;\varepsilon_1,\dots,\varepsilon_n)$ be an naive EGK datum of length n, and $n_1,\dots,n_r,n_1^*,\dots,n_r^*$ and $m_1,\dots,m_r$ be those defined in Definition 3.1. Then put $\zeta_s = \varepsilon_{n_s^*}$ for $s = 1,\dots,r$. Then $G_H := (n_1,\dots,n_r;m_1,\dots,m_r;\zeta_1,\dots,\zeta_r)$ is an EGK datum (cf. [[4], Proposition 6.2]). We then define a mapping Υ_n from $\mathcal{NEGK}_n$ to $\mathcal{EGK}_n$ by $\Upsilon_n(H) = G_H$. Then we easily see that The mapping Υ_n is surjective (cf. [[4], Proposition 6.3]). We note that Υ_n is not injective in general.

Theorem 4.3. *Let G be an* EGK *datum of length n, take $H \in \mathcal{NEGK}_n$ such that $\Upsilon_n(H) = G$. Then $\mathcal{F}(H;Y,X)$ is uniquely determined by G, and does not depend on the choice of H.*

For an EGK datum G we define $\widetilde{\mathcal{F}}(G;Y,X)$ as $\mathcal{F}(H;Y,X)$, where H is a naive EGK datum of length n such that $\Upsilon_n(H) = G$.

5 Main result

Now we state our main result.

Theorem 5.1. *Let B be a non-degenerate half-integral matrix of degree n over $\mathfrak{o}$. Then we have*

$$\widetilde{F}(B,X) = \widetilde{\mathcal{F}}(\mathrm{EGK}(B);q^{1/2},X).$$

In particular, $\widetilde{F}(B,X)$ is determined by EGK(B).

We give an outline of the proof. First assume that q is odd. We may assume that $B \in \mathcal{H}_n(\mathfrak{o})$ is a diagonal Jordan form with $\mathrm{GK}(B) = (a_1,\dots,a_n)$. Then $B^{(n-1)}$ is also a diagonal Jordan form with $\mathrm{GK}(B) = (a_1,\dots,a_{n-1})$ if $n \geq 2$. Then Theorem 5.1 follows from the following induction formulas.

Theorem 5.2. *Under the above notation and the assumption, we have the following.*

(1) Let $n = 1$. Then

$$\widetilde{F}(B,X) = \sum_{i=0}^{a_1} X^{i-(a_1/2)}$$

(2) Let $n \geq 3$. Then

$$\widetilde{F}(B,X) = D(\mathfrak{e}_n,\mathfrak{e}_{n-1},\xi_{B^{(n-1)}};X)\widetilde{F}(B^{(n-1)},q^{1/2}X)$$
$$+\,\eta_B D(\mathfrak{e}_n,\mathfrak{e}_{n-1},\xi_{B^{(n-1)}};X^{-1})\widetilde{F}(B^{(n-1)},q^{1/2}X^{-1})$$

if n is odd, and

$$\widetilde{F}(B,X) = C(\mathfrak{e}_n,\mathfrak{e}_{n-1},\xi_B;X)\widetilde{F}(B^{(n-1)},q^{1/2}X)$$
$$+\,C(\mathfrak{e}_n,\mathfrak{e}_{n-1},\xi_B;X^{-1})\widetilde{F}(B^{(n-1)},q^{1/2}X^{-1})$$

if n is even.

Next we consider a more complicated case where q is even. Let B be a reduced form in $\mathcal{H}_n(\mathfrak{o})$ with GK-type $((a_1,\dots,a_n),\sigma)$. Put $\underline{a} = (a_1,\dots,a_n)$. We say that $(\underline{a},\sigma)$ belongs to category (I) if $n = \sigma(n-1)$ and $a_{n-1} = a_n$. We say that σ belongs to category (II) if B does not belong to category (I). We note that $(\underline{a},\sigma)$ belongs to category (II) if and only if $a_{n-1} < a_n$ or $\sigma(n) = n$. In particular, $(\underline{a},\sigma)$ belongs to category (II) if $n = 1$. We also say that B belongs to category (I) or (II) according as $(\underline{a},\sigma)$ belongs to category (I) or (II). We note that if two reduced forms are of the same GK-type, then they belong to the same category. Let $B = (b_{ij}) \in \mathcal{H}_n(\mathfrak{o})^{\mathrm{nd}}$ be a reduced form of type $(\underline{a}, \sigma)$. Put $\underline{a} = (a_1, \ldots, a_n)$. For a non-negative integer $i \le n$ let $\mathfrak{e}_i = \mathfrak{e}(\underline{a})_i$ be the integer in Definition 4.3. By [[4], Theorem 0.1], we have $\mathfrak{e}_B = \mathfrak{e}_n$. Now we prove the following assertion by induction on n:

$$\widetilde{F}(B, X) = \widetilde{\mathcal{F}}(\mathrm{EGK}(B); q^{1/2}, X). \qquad (\mathrm{EF}_n)$$

First we easily see that the following.

Proposition 5.1. *Let $n = 1$. Then, we have*

$$\widetilde{F}(B, X) = \sum_{i=0}^{a_1} X^{i-(a_1/2)}.$$

Next let us consider the case where $n \ge 2$. We give induction formulas for $\widetilde{F}(B, X)$ in the following cases, which proves Theorem 5.1 combined with Definition 4.3 and Proposition 4.2.

Case 1. Assume that B satisfies either one of the following conditions:
(1) B belongs to category (II)
(2) B belongs to category (I) and $n + a_1 + \cdots + a_{2[n/2]}$ is even.
If B satisfies the condition (1), then $B^{(n-1)}$ is a reduced form with $\mathrm{GK}(B^{(n-1)}) = \underline{a}^{(n-1)}$. If B satisfies the condition (2), then we easily see that B is $GL_n(\mathfrak{o})$-equivalent to a reduced form $\widetilde{B}$ such that $\widetilde{B}^{(n-1)}$ is a reduced form with $\mathrm{GK}(\widetilde{B}^{(n-1)}) = \underline{a}^{(n-1)}$. Therefore, we may assume that $B^{(n-1)}$ is a reduced form with $\mathrm{GK}(B^{(n-1)}) = \underline{a}^{(n-1)}$.

Theorem 5.3. *Let the notation and the assumption be as above. Then, under the assumption* EF_{n-1}*, we have*

$$\begin{aligned}\widetilde{F}(B, X) =& D(\mathfrak{e}_n, \mathfrak{e}_{n-1}, \xi_{B^{(n-1)}}; X)\widetilde{F}(B^{(n-1)}, q^{1/2}X)\\ &+ \eta_B D(\mathfrak{e}_n, \mathfrak{e}_{n-1}, \xi_{B^{(n-1)}}; X^{-1})\widetilde{F}(B^{(n-1)}, q^{1/2}X).\end{aligned}$$

if n is odd, and

$$\begin{aligned}\widetilde{F}(B, X) =& C(\mathfrak{e}_n, \mathfrak{e}_{n-1}, \xi_B; X)\widetilde{F}(B^{(n-1)}, q^{1/2}X)\\ &+ C(\mathfrak{e}_n, \mathfrak{e}_{n-1}, \xi_B; X^{-1})\widetilde{F}(B^{(n-1)}, q^{1/2}X)\end{aligned}$$

if n is even.

Case 2. Assume that B belongs to category (I) and that $n + a_1 + \cdots + a_{n-1}$ is odd. Then, $B^{(n-2)}$ is a reduced form with $\mathrm{GK}(B^{(n-2)}) = \underline{a}^{(n-2)}$.

Theorem 5.4. *Let the notation and the assumption be as above. Then under the assumption* EF_{n-2}*, we have the following.*

(1) Assume that n is odd and that $a_1+\cdots+a_{n-1}$ is even. Then we have

$$\begin{aligned}\widetilde{F}(B,X) =&q^{\mathfrak{e}_{n-2}/2-1/2}\left\{\frac{X^{(-\mathfrak{e}_n+\mathfrak{e}_{n-2})/2-1}}{(q^{1/2}X)^{-1}-q^{1/2}X}\widetilde{F}(B^{(n-2)},qX)\right.\\ &\left.+\eta_B\frac{X^{(\mathfrak{e}_n-\mathfrak{e}_{n-2})/2+1}}{(q^{1/2}X^{-1})^{-1}-q^{1/2}X^{-1}}\widetilde{F}(B^{(n-2)},qX^{-1})\right\}\\ &+\eta_B\frac{q^{\mathfrak{e}_{n-1}/2}(q-q^{-1})}{((q^{1/2}X)^{-1}-q^{1/2}X)((q^{1/2}X^{-1})^{-1}-q^{1/2}X^{-1})}\\ &\times\widetilde{F}(B^{(n-2)},X).\end{aligned}$$

(2) Let n be even and that $a_1+\cdots+a_{n-2}$ is odd. Then

$$\begin{aligned}\widetilde{F}(B,X) = q^{\mathfrak{e}_{n-2}/2}&\left\{\frac{X^{(-\mathfrak{e}_n+\mathfrak{e}_{n-2})/2-1}}{X^{-1}-X}\widetilde{F}(B^{(n-2)},qX)\right.\\ &\left.+\frac{X^{(\mathfrak{e}_n-\mathfrak{e}_{n-2})/2+1}}{X-X^{-1}}\widetilde{F}(B^{(n-2)},qX^{-1})\right\}.\end{aligned}$$

6 Examples

(1) Let $G=(n_1,\ldots,n_r;m_1,\ldots,m_r;\zeta_1,\ldots,\zeta_r)$ be an EGK datum of length n. For $1\le i\le n$ we define $\widetilde{m}_i$ as

$$\widetilde{m}_i=m_j \text{ if } n_1+\cdots+n_{j-1}+1\le i\le n_1+\cdots+n_j,$$

and for such $\widetilde{m}_1,\ldots,\widetilde{m}_n$ we define the integers $\mathfrak{e}_1,\ldots,\mathfrak{e}_n$ as in Definition 4.3.

(1.1) An EGK datum of length 2 is one of the following forms

(a) $G=(1,1;m_1,m_2;1,\zeta_2)$ with $m_1<m_2$ and $\zeta_2\in\mathcal{Z}_3$

(b) $G=(2;m_1;\zeta_1)$ with $\zeta_2\in\{\pm1\}$.

Put $\xi=\zeta_2$ or $\xi=\zeta_1$ according as case (a) or case (b). Then

$$H=(\widetilde{m}_1,\widetilde{m}_2;1,\xi)$$

is a naive EGK datum such that $\Upsilon_2(H)=G$, and by a simple computation (cf.[[5], Corollary 4.1]), $\widetilde{\mathcal{F}}(G;Y,X)$ can be expressed as

$$\begin{aligned}&\widetilde{\mathcal{F}}(G;Y,X)\\ &=\sum_{i=0}^{\mathfrak{e}_1}Y^i\left\{\frac{X^{-\mathfrak{e}_2/2+i-1}-X^{\mathfrak{e}_2/2-i+1}}{X^{-1}-X}\right\}-\xi\sum_{i=0}^{\mathfrak{e}_1}Y^{i-1}\left\{\frac{X^{-\mathfrak{e}_2/2+i}-X^{\mathfrak{e}_2/2-i}}{X^{-1}-X}\right\}.\end{aligned}$$

Let $B\in\mathcal{H}_2(\mathfrak{o})^{\mathrm{nd}}$. Then by Theorem 5.1, we have

$$\widetilde{F}(B,X)=\widetilde{\mathcal{F}}(\mathrm{EGK}(B);q^{1/2},X).$$

This coincides with [[9], Corollary 5.1].

(1.2) An EGK datum of length 3 is one of the following forms:

(a) $G = (1,1,1;m_1,m_2,m_3;1,\zeta_2,\zeta_3)$ with $\zeta_2 \in \mathcal{Z}_3$, and $\zeta_3 \in \{\pm 1\}$

(b) $G = (1,2;m_1,m_2;1,\zeta_2)$ with $\zeta_2 \in \{\pm 1\}$

(c) $G = (2,1;m_1,m_2;\zeta_1,\zeta_2)$ with $\zeta_1 \in \mathcal{Z}_3$ and $\zeta_2 \in \{\pm 1\}$

(d) $G = (3;m_1;1)$.

We put

$$\xi = \begin{cases} \zeta_2 & \text{in case (a)} \\ \zeta_1 & \text{in case (c)} \\ 1 & \text{in case (b) or case (d), and } \widetilde{m}_1 + \widetilde{m}_2 \text{ is even} \\ 0 & \text{in case (b) or case (d), and } \widetilde{m}_1 + \widetilde{m}_2 \text{ is odd,} \end{cases}$$

and

$$\eta = \begin{cases} \zeta_3 & \text{in case (a)} \\ \zeta_2 & \text{in case (b) or (c)} \\ 1 & \text{in case (d).} \end{cases}$$

Moreover let $\mathfrak{e}_2' = 2[(a_1 + a_2 + a_3 + 1)/2]$. Then,

$$H = (\widetilde{m}_1, \widetilde{m}_2, \widetilde{m}_3; 1, \xi, \eta)$$

is a naive EGK datum such that $\Upsilon_3(H) = G$, and by a simple computation(cf.[[5], Corollary 4.1]), $\widetilde{\mathcal{F}}(G;Y,X)$ can be expressed as

$$\begin{aligned}
&\widetilde{\mathcal{F}}(G;Y,X) = X^{-\mathfrak{e}_3/2} \\
&\times \left\{ \sum_{i=0}^{\mathfrak{e}_1} (Y^2X)^i \sum_{j=0}^{\mathfrak{e}_2'/2-i-1} (YX)^{2j} + \eta X^{\mathfrak{e}_3} \sum_{i=0}^{\mathfrak{e}_1} (Y^2X^{-1})^i \sum_{j=0}^{\mathfrak{e}_2'/2-i-1} (YX^{-1})^{2j} \right. \\
&\left. + \xi^2 Y^{\mathfrak{e}_2'} X^{\mathfrak{e}_2'-\mathfrak{e}_1} \sum_{j=0}^{\mathfrak{e}_3-2\mathfrak{e}_2'+\mathfrak{e}_1} (\xi X)^j \sum_{i=0}^{\mathfrak{e}_1} X^i \right\}.
\end{aligned}$$

Let $B \in \mathcal{H}_3(\mathfrak{o})^{\mathrm{nd}}$. Then by Theorem 5.1, we have

$$\widetilde{F}(B,X) = \widetilde{\mathcal{F}}(\mathrm{EGK}(B); q^{1/2}, X).$$

This essentially coincides with [[7], Example (3)] and [[11], (2.8)] in the case $F = \mathbb{Q}_p$.
(2) Let q be odd, and let

$$B \sim \varpi^{a_1} u_1 \perp \cdots \perp \varpi^{a_n} u_n \quad (a_1 \leq \cdots \leq a_n,\ u_1, \ldots, u_n \in \mathfrak{o}^\times)$$

be a diagonal Jordan decomposition of $B \in \mathcal{H}_n(\mathfrak{o})^{\mathrm{nd}}$. Put

$$\varepsilon_i = \begin{cases} \xi_{B^{(i)}} & \text{if } i \text{ is even} \\ \eta_{B^{(i)}} & \text{if } i \text{ is odd.} \end{cases}$$

Then $H = (a_1, \dots, a_n; \varepsilon_1, \dots, \varepsilon_n)$ is a naive EGK datum such that $\Upsilon_n(H) = \mathrm{EGK}(B)$, and by Proposition 4.1 and Theorem 5.1, we can get an explicit formula for $\widetilde{F}(B, X)$ in terms of H, which is essentially coincides with [[7], Theorem 4.3] in the case where $F = \mathbb{Q}_p$. In the dyadic case, if one can get a naive *EGK* datum associated with $B \in \mathcal{H}_n(\mathfrak{o})^{\mathrm{nd}}$, we can also give an explicit formula for $\widetilde{F}(B, X)$ in terms of it.

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